Ain Shams Engineering Journal (2013) **4**, 919–924

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Integral representations for the product of certain polynomials of two variables

Mumtaz Ahmad Khan, Abdul Hakim Khan, Sayed Mohammad Abbas *

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202 002, UP, India

Received 20 December 2012; accepted 8 February 2013
Available online 3 April 2013

KEYWORDS

Laguerre polynomials of two variables;
Jacobi polynomials of two variables;
Generalized Bessel polynomials of two variables;
Generalized Rice polynomials of two variables;
Krawtchouk polynomials of two variables;
Meixner polynomials of two variables;
Gottlieb polynomials of two variables;
Poisson–Charlier polynomials of two variables

Abstract The main object of this paper is to investigate several integral representations for the product of two polynomials of two variables, e.g. Laguerre, Jacobi, Generalized Bessel, Generalized Rice, Krawtchouk, Meixner, Gottlieb and Poisson–Charlier polynomials of two variables.

1. Introduction

In 1938, Watson [1] gave the integral representation for the product $L_m^{(\alpha)}(x)L_n^{(\beta)}(y)$, which was generalized by Carlitz [2] in the form

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = \frac{2^{\alpha+\beta+m+n}}{\pi^2}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\phi i+(\alpha-\beta)\theta i}\cos^{m+n}\phi\cos^{\alpha+\beta}\theta \times L_{m+n}^{(\alpha+\beta)}\left(\frac{xe^{(\theta-\phi)i}+ye^{-(\theta-\phi)i}}{\cos\phi}\cos\theta\right)d\phi d\theta, \quad (1.1)$$

where $L_n^{(\alpha)}(x)$ denotes the general Laguerre polynomials of degree n.

Following the method adopted by Carlitz [2], Chatterjea [3] gave the integral representation for the product of two generalized Bessel polynomials. Further, in 1964, Chatterjea [4] gave

* Corresponding author. Tel.: +91 9758146013.
E-mail address: smabbas.alig@gmail.com (S.M. Abbas).
Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

2090-4479 © 2013 Ain Shams University. Production and hosting by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.asej.2013.02.001

the integral formula for the product of two Jacobi polynomials. Also, in 1969, Manocha [5] gave the integral representation for the product of two generalized Rice polynomials.

In 1976, Srivastava and Panda [6] derived an integral representation for the product of two Jacobi polynomials and also gave some generalization involving Kampé de Fériet's double hypergeometric functions. Also in 2004, Lin et al. [7] derived integral representation for the product of two polynomials of the classes defined by (3)–(6) in [7].

Recently in 2011, Khan et al. [8] gave the integral representation for the product of several other two polynomials, e.g. Meixner, Krawtchouk, Gottlieb and Poisson–Charlier polynomials of one variable.

Motivated by the above work the present paper deals with several integral representations for the product of two polynomials of two variables.

In the present paper we need the following definitions of two variables polynomials:

The Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ are defined as:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}}{r!s!(1+\alpha)_r(1+\beta)_s}x^ry^s. \tag{1.2}$$

The Jacobi polynomials of two variables $P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,y)$ are defined as:

$$P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,y)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(1+\alpha_2+\beta_2+n)_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s}\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s. \tag{1.3}$$

The Rice polynomials of two variables $H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(\xi_1,\xi_2,p_1,p_2,x,y)$ are defined as:

$$H_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(\xi_1,\xi_2,p_1,p_2,x,y)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+n)_r(\xi_1)_r(1+\alpha_2+\beta_2+n)_s(\xi_2)_s}{r!s!(1+\alpha_1)_r(p_1)_r(1+\alpha_2)_s(p_2)_s}x^ry^s. \tag{1.4}$$

The Bessel polynomials of two variables $Y_n(x, a_1, b_1; y, a_2, b_2)$ are defined as:

$$Y_n(x,a_1,b_1;y,a_2,b_2)=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(a_1-1+n)_r(a_2-1+n)_s}{r!s!}\left(-\frac{x}{b_1}\right)^r\left(-\frac{y}{b_2}\right)^s. \tag{1.5}$$

The Krawtchouk polynomials of two variables $K_n(x;\lambda_1,N_1:y;\lambda_2,N_2)$ are defined as:

$$K_n(x;\lambda_1,N_1:y;\lambda_2,N_2)=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(-x)_r(-y)_s}{r!s!(-N_1)_r(-N_2)_s}\left(\frac{1}{\lambda_1}\right)^r\left(\frac{1}{\lambda_2}\right)^s. \tag{1.6}$$

The Meixner polynomials of two variables $M_n(x;\beta_1,\lambda_1:y;\beta_2,\lambda_2)$ are defined as:

$$M_n(x;\beta_1,\lambda_1:y;\beta_2,\lambda_2)=(\beta_1)_n(\beta_2)_n\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(-x)_r(-y)_s}{r!s!(\beta_1)_r(\beta_2)_s}\left(1-\frac{1}{\lambda_1}\right)^r\left(1-\frac{1}{\lambda_2}\right)^s. \tag{1.7}$$

The Gottlieb polynomials of two variables $I_n(x;\lambda:y;\mu)$ are defined as:

$$I_n(x;\lambda:y;\mu)=e^{-n(\lambda+\mu)}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(-x)_r(-y)_s}{r!s!(1)_{r+s}}(1-e^{\lambda})^r(1-e^{\mu})^s. \tag{1.8}$$

The Poisson–Charlier polynomials of two variables $C_n(x;\alpha_1:y;\alpha_2)$ are defined as:

$$C_n(x;\alpha_1:y;\alpha_2)=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(-x)_r(-y)_s}{r!s!}\left(-\frac{1}{\alpha_1}\right)^r\left(-\frac{1}{\alpha_2}\right)^s. \tag{1.9}$$

2. Integral representations for the product of two variables polynomials

For the polynomials $L_n^{(\alpha,\beta)}(x,y)$ defined by (1.2), we begin by considering the following product:

$$\begin{aligned}L_m^{(\alpha,\beta)}(x,y)L_n^{(\gamma,\delta)}(u,v)&=\frac{(1+\alpha)_m(1+\beta)_m(1+\gamma)_n(1+\delta)_n}{(m!)^2(n!)^2}\\&\times\sum_{r=0}^{m}\sum_{s=0}^{m-r}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-m)_{r+s}(-n)_{k+l}}{r!s!k!l!(1+\alpha)_r(1+\beta)_s(1+\gamma)_k(1+\delta)_l}x^ry^su^kv^l\\&=\frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\gamma+n+1)\Gamma(\delta+n+1)}{m!n!}\\&\times\sum_{r=0}^{m}\sum_{k=0}^{n}\sum_{s=0}^{m-r}\sum_{l=0}^{n-k}\frac{\Gamma(m+n-r-s-k-l+1)}{\Gamma(m-r-s+1)\Gamma(n-k-l+1)}\frac{\Gamma(\alpha+\gamma+r+k+1)}{\Gamma(\alpha+r+1)\Gamma(\gamma+k+1)}\\&\times\frac{\Gamma(\beta+\delta+s+l+1)}{\Gamma(\beta+s+1)\Gamma(\delta+l+1)}\frac{(-x)^r(-y)^s(-u)^k(-v)^l}{r!s!k!l!\Gamma(m+n-r-s-k-l+1)}\times\frac{1}{\Gamma(\alpha+\gamma+r+k+1)\Gamma(\beta+\delta+s+l+1)}.\end{aligned} \tag{2.1}$$

Now we notice the results [4]

$$\frac{\Gamma(\mu+v+1)}{\Gamma(\mu+1)(v+1)}=\frac{2^{\mu+v}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{(\mu-v)\theta i}\cos^{\mu+v}\theta d\theta,\quad(\mu+v>-1) \tag{2.2}$$

and

$$\frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)} = \int_0^1 t^{\mu-1}(1-t)^{v-1}dt, \quad (\mu > 0, v > 0). \tag{2.3}$$

It therefore follows from (2.1) that

$$\begin{aligned} L_m^{(\alpha,\beta)}(x,y)L_n^{(\gamma,\delta)}(u,v) = {} & \frac{2^{\alpha+\beta+\gamma+\delta+m+n}}{\pi^3}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\gamma+n+1)\Gamma(\delta+n+1)}{m!n!(m+n)!\Gamma(\alpha+\gamma+1)\Gamma(\beta+\delta+1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\psi i+(\alpha-\gamma)\phi i+(\beta-\delta)\theta i} \\ & \times \cos^{m+n}\psi\cos^{\alpha+\gamma}\phi\cos^{\beta+\delta}\theta \times \sum_{r=0}^{m}\sum_{k=0}^{n}\sum_{s=0}^{m-r}\sum_{l=0}^{n-k}\frac{(-m-n)_{r+k+s+l}}{r!k!s!l!(1+\alpha+\gamma)_{r+k}(1+\beta+\delta)_{s+l}}\left(xe^{-(\psi-\phi)i}\frac{\cos\phi}{\cos\psi}\right)^r \\ & \times \left(e^{(\psi-\phi)i}\frac{\cos\phi}{\cos\psi}\right)^k\left(ye^{-(\psi-\theta)i}\frac{\cos\theta}{\cos\psi}\right)^s\left(ve^{(\psi-\theta)i}\frac{\cos\theta}{\cos\psi}\right)^l d\psi d\phi d\theta. \end{aligned} \tag{2.4}$$

Since

$$\sum_{m,n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}, \tag{2.5}$$

so that

$$\begin{aligned} L_m^{(\alpha,\beta)}(x,y)L_n^{(\gamma,\delta)}(u,v) = {} & \frac{2^{\alpha+\beta+\gamma+\delta+m+n}}{\pi^3}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\gamma+n+1)\Gamma(\delta+n+1)}{m!n!(m+n)!\Gamma(\alpha+\gamma+1)\Gamma(\beta+\delta+1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\psi i+(\alpha-\gamma)\phi i+(\beta-\delta)\theta i}\cos^{m+n}\psi\cos^{\alpha+\gamma}\phi\cos^{\beta+\delta}\theta \\ & \times \sum_{M=0}^{m+n}\sum_{N=0}^{(m+n)-M}\frac{(-m-n)_{M+N}}{M!N!(1+\alpha+\gamma)_M(1+\beta+\delta)_N}\left\{\frac{xe^{(\phi-\psi)i}+ue^{-(\phi-\psi)i}}{\cos\psi}\cos\phi\right\}^M \times \left\{\frac{ye^{(\theta-\psi)i}+ve^{-(\theta-\psi)i}}{\cos\psi}\cos\theta\right\}^N d\psi d\phi d\theta, \end{aligned} \tag{2.6}$$

which by virtue of (1.2), yields

$$\begin{aligned} L_m^{(\alpha,\beta)}(x,y)L_n^{(\gamma,\delta)}(u,v) = {} & \frac{2^{\alpha+\beta+\gamma+\delta+m+n}}{\pi^3}\frac{(m+n)!}{m!n!} \times \frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\gamma+n+1)\Gamma(\delta+n+1)}{\Gamma(\alpha+\gamma+m+n+1)\Gamma(\beta+\delta+m+n+1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ & \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\psi i+(\alpha-\gamma)\phi i+(\beta-\delta)\theta i}\cos^{m+n}\psi\cos^{\alpha+\gamma}\phi\cos^{\beta+\delta}\theta \\ & \times L_{m+n}^{(\alpha+\gamma,\beta+\delta)}\left(\frac{xe^{(\phi-\psi)i}+ue^{-(\phi-\psi)i}}{\cos\psi}\cos\phi, \frac{ye^{(\theta-\psi)i}+ve^{-(\theta-\psi)i}}{\cos\psi}\cos\theta\right)d\psi d\phi d\theta. \end{aligned} \tag{2.7}$$

For $\alpha = \beta = \gamma = \delta = 0$, (2.7) reduces to

$$L_m(x,y)L_n(u,v) = \frac{2^{m+n}}{\pi^3}\frac{m!n!}{(m+n)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\psi i}\cos^{m+n}\psi \times L_{m+n}\left(\frac{xe^{(\phi-\psi)i}+ue^{-(\phi-\psi)i}}{\cos\psi}\cos\phi, \frac{ye^{(\theta-\psi)i}+ve^{-(\theta-\psi)i}}{\cos\psi}\cos\theta\right)d\psi d\phi d\theta. \tag{2.8}$$

In a similar manner, we can derive the following integral representations for the product of two polynomials of two variables defined by (1.3)–(1.9):

$$\begin{aligned}
&P_m^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(x,y)P_n^{(\gamma_1,\delta_1;\gamma_2,\delta_2)}(u,v) = \frac{2^{\alpha_1+\alpha_2+\gamma_1+\gamma_2+m+n}}{\pi^3}\frac{(m+n)!}{m!n!}\\
&\times\frac{\Gamma(\alpha_1+m+1)\Gamma(\alpha_2+m+1)\Gamma(\gamma_1+n+1)\Gamma(\gamma_2+n+1)}{\Gamma(\alpha_1+\beta_1+m+1)\Gamma(\alpha_2+\beta_2+m+1)\Gamma(\gamma_1+\delta_1+n+1)\Gamma(\gamma_2+\delta_2+n+1)}\\
&\times\frac{\Gamma(\alpha_1+\beta_1+\gamma_1+\delta_1+m+n+2)\Gamma(\alpha_2+\beta_2+\gamma_2+\delta_2+m+n+2)}{\Gamma(\alpha_1+\gamma_1+m+n+1)\Gamma(\alpha_2+\gamma_2+m+n+1)}\\
&\times\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{\alpha_1+\beta_1+m}z^{\alpha_2+\beta_2+m}(1-t)^{\gamma_1+\delta_1+n}(1-z)^{\gamma_2+\delta_2+n}e^{(m-n)\psi i+(\alpha_1-\gamma_1)\phi i+(\alpha_2-\gamma_2)\theta i}\\
&\times\cos^{m+n}\psi\cos^{\alpha_1+\gamma_1}\phi\cos^{\alpha_2+\gamma_2}\theta\times P_{m+n}^{(\alpha_1+\gamma_1,\beta_1+\delta_1+1;\alpha_2+\gamma_2,\beta_2+\delta_2+1)}\\
&\times\left(1-\frac{(1-x)te^{(\phi-\psi)i}+(1-u)(1-t)e^{-(\phi-\psi)i}}{\cos\psi}\cos\phi, 1-\frac{(1-y)ze^{(\theta-\psi)i}+(1-v)(1-z)e^{-(\theta-\psi)i}}{\cos\psi}\cos\theta\right)d\psi d\phi d\theta dz dt,
\end{aligned}\tag{2.9}$$

As a particular case of (2.9) we note that

$$\begin{aligned}
P_m(x,y)P_n(u,v) = {}&\frac{2^{m+n}}{\pi^3}\frac{(m+n+1)\Gamma(m+n+2)}{m!n!}\times\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^m z^m(1-t)^n(1-z)^n e^{(m-n)\psi i}\cos^{m+n}\psi\\
&\times P_{m+n}^{(0,1;0,1)}\left(1-\frac{(1-x)te^{(\phi-\psi)i}+(1-u)(1-t)e^{-(\phi-\psi)i}}{\cos\psi}\cos\phi, 1-\frac{(1-y)ze^{(\theta-\psi)i}+(1-v)(1-z)e^{-(\theta-\psi)i}}{\cos\psi}\cos\theta\right)d\psi d\phi d\theta dz dt,
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
&H_m^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(\xi_1,\xi_2,p_1,p_2,x,y)H_n^{(\gamma_1,\delta_1;\gamma_2,\delta_2)}(\eta_1,\eta_2,q_1,q_2,u,v)\\
&=\frac{2^{\alpha_1+\alpha_2+\gamma_1+\gamma_2+p_1+p_2+q_1+q_2+m+n-4}}{\pi^5}\frac{(m+n)!}{m!n!}\frac{\Gamma(\alpha_1+m+1)\Gamma(\alpha_2+m+1)}{\Gamma(\alpha_1+\beta_1+m+1)\Gamma(\alpha_2+\beta_2+m+1)}\\
&\times\frac{\Gamma(\gamma_1+n+1)\Gamma(\gamma_2+n+1)\Gamma(\alpha_1+\beta_1+\gamma_1+\delta_1+m+n+2)}{\Gamma(\gamma_1+\delta_1+n+1)\Gamma(\gamma_2+\delta_2+n+1)\Gamma(\alpha_1+\gamma_1+m+n+1)\Gamma(\alpha_2+\gamma_2+m+n+1)}\\
&\times\frac{\Gamma(\alpha_2+\beta_2+\gamma_2+\delta_2+m+n+2)\Gamma(\xi_1+\eta_1)\Gamma(\xi_2+\eta_2)\Gamma(p_1)\Gamma(p_2)\Gamma(q_1)\Gamma(q_2)}{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(p_1+q_1-1)\Gamma(p_2+q_2-1)}\\
&\times\int_0^1\int_0^1\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s^{\alpha_1+\beta_1+m}t^{\alpha_2+\beta_2+m}w^{\xi_1-1}z^{\xi_2-1}(1-s)^{\gamma_1+\delta_1+n}(1-t)^{\gamma_2+\delta_2+n}(1-w)^{\eta_1-1}\\
&\times(1-z)^{\eta_2-1}e^{(m-n)\omega i+(\alpha_1-\gamma_1)\psi i+(\alpha_2-\gamma_2)\varphi i+(p_1-q_1)\phi i+(p_2-q_2)\theta i}\cos^{m+n}\omega\cos^{\alpha_1+\gamma_1}\psi\cos^{\alpha_2+\gamma_2}\varphi\\
&\times\cos^{p_1+q_1-2}\phi\cos^{p_2+q_2-2}\theta H_{m+n}^{(\alpha_1+\gamma_1,\beta_1+\delta_1+1;\alpha_2+\gamma_2,\beta_2+\delta_2+1)}\Big(\xi_1+\eta_1,\xi_2+\eta_2,p_1+q_1-1,p_2+q_2-1,\frac{4(xswe^{(\phi+\psi-\omega)i}+u(1-s)(1-w)e^{-(\phi+\psi-\omega)i})}{\cos\omega}\\
&\cos\phi\cos\psi,\frac{4(ytze^{(\theta+\varphi-\omega)i}+v(1-t)(1-z)e^{-(\theta+\varphi-\omega)i})}{\cos\omega}\cos\theta\cos\varphi\Big)d\omega d\psi d\varphi d\phi d\theta ds dt dw dz,
\end{aligned}\tag{2.11}$$

As a particular case of (2.11) we note that

$$
\begin{aligned}
H_m(\xi_1,\xi_2,p_1,p_2,x,y)H_n(\eta_1,\eta_2,q_1,q_2,u,v) &= \frac{2^{p_1+p_2+q_1+q_2+m+n-4}}{\pi^5}\times\frac{(m+n+1)^2(m+n)!}{m!n!}\frac{\Gamma(\xi_1+\eta_1)\Gamma(\xi_2+\eta_2)\Gamma(p_1)\Gamma(p_2)\Gamma(q_1)\Gamma(q_2)}{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(p_1+q_1-1)\Gamma(p_2+q_2-1)}\\
&\times\int_0^1\int_0^1\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s^m t^m w^{\xi_1-1}z^{\xi_2-1}(1-s)^n(1-t)^n(1-w)^{\eta_1-1}(1-z)^{\eta_2-1}\\
&\times e^{(m-n)\omega i+(p_1-q_1)\phi i+(p_2-q_2)\theta i}\cos^{m+n}\omega\cos^{p_1+q_1-2}\phi\cos^{p_2+q_2-2}\theta\times H_{m+n}^{(0,1;0,1)}\\
&\times\left(\xi_1+\eta_1,\xi_2+\eta_2,p_1+q_1-1,p_2+q_2-1,\frac{4(xswe^{(\phi+\psi-\omega)i}+u(1-s)(1-w)e^{-(\phi+\psi-\omega)i})}{\cos\omega}\right.\\
&\left.\times\cos\phi\cos\psi,\frac{4(ytze^{(\theta+\varphi-\omega)i}+v(1-t)(1-z)e^{-(\theta+\varphi-\omega)i})}{\cos\omega}\cos\theta\cos\varphi\right)d\omega d\psi d\varphi d\phi d\theta dsdtdwdz, \quad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
Y_m(x,a_1,b_1;y,a_2,b_2)Y_n(u,c_1,d_1;v,c_2,d_2) &= \frac{2^{m+n}}{\pi}\frac{m!n!}{(m+n)!}\times\frac{\Gamma(a_1+c_1+m+n-2)\Gamma(a_2+c_2+m+n-2)}{\Gamma(a_1+m-1)\Gamma(a_2+m-1)\Gamma(c_1+n-1)\Gamma(c_2+n-1)}\times\int_0^1\int_0^1\\
&\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{a_1+m-2}z^{a_2+m-2}(1-t)^{c_1+n-2}(1-z)^{c_2+n-2}e^{(m-n)\theta i}\cos^{m+n}\theta\\
&\times Y_{m+n}\left(\frac{d_1xte^{-i\theta}+b_1u(1-t)e^{i\theta}}{2\cos\theta},a_1+c_1-1,b_1d_1;\frac{d_2yze^{-i\theta}+b_2v(1-z)e^{i\theta}}{2\cos\theta},a_2+c_2-1,b_2d_2\right)d\theta dtdz,
\end{aligned}
\quad (2.13)
$$

$$
\begin{aligned}
&K_m(x;\lambda_1,M_1:y;\lambda_2,M_2)K_n(u;\mu_1,N_1:v;\mu_2,N_2)\\
&\quad=\frac{2^{x+y+u++m+n}}{\pi^3}\frac{m!n!x!y!u!!}{(m+n)!(x+u)!(y+)!}\times\frac{(M_1+N_1+1)!(M_2+N_2+1)!}{M_1!M_2!N_1!N_2!}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{M_1}z^{M2}(1-t)^{N_1}(1-z)^{N_2}e^{(m-n)\psi i+(x-u)\phi i+(y-v)\theta i}\\
&\quad\times\cos^{m+n}\psi\cos^{x+u}\phi\cos^{y+v}\theta K_{m+n}\left(x+u;\frac{4\lambda_1\mu_1t(1-t)\cos\phi\cos\psi}{\lambda_1te^{(\phi+\psi)i}+\mu_1(1-t)e^{-(\phi+\psi)i}},M_1+N_1+1:y+v;\frac{4\lambda_2\mu_2z(1-z)\cos\theta\cos\psi}{\lambda_2ze^{(\theta+\psi)i}+\mu_2(1-z)e^{-(\theta+\psi)i}},M_2+N_2+1\right)d\psi d\phi d\theta dtdz,
\end{aligned}
\quad (2.14)
$$

$$
\begin{aligned}
&M_m(x;\beta_1,\lambda_1:y;\beta_2,\lambda_2)M_n(u;\gamma_1,\mu_1:v;\gamma_2,\mu_2)=\frac{2^{x+y+u+v+\beta_1+\beta_2+\gamma_1+\gamma_2+m+n-4}}{\pi^5}\\
&\quad\times\frac{m!n!x!y!u!v!}{(m+n)!(x+u)!(y+v)!}\frac{\Gamma(\beta_1+m)\Gamma(\beta_2+m)\Gamma(\gamma_1+n)\Gamma(\gamma_2+n)}{\Gamma(\beta_1+\gamma_1+m+n-1)\Gamma(\beta_2+\gamma_2+m+n-1)}\\
&\quad\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\omega i+(x-u)\psi i+(y-v)\varphi i+(\beta_1-\gamma_1)\phi i+(\beta_2-\gamma_2)\theta i}\cos^{m+n}\omega\cos^{x+u}\psi\cos^{y+v}\varphi\times\cos^{\beta_1+\gamma_1-2}\phi\cos^{\beta_2+\gamma_2-2}\theta M_{m+n}\\
&\quad\left(x+u;\beta_1+\gamma_1-1,\frac{2\lambda_1\mu_1\cos\psi\cos\omega}{2\lambda_1\mu_1\cos\psi\cos\omega-\{\mu_1(\lambda_1-1)e^{(\phi-\psi-\omega)i}+\lambda_1(\mu_1-1)e^{-(\phi-\psi-\omega)i}\}\cos\phi}:\right.\\
&\quad\left.y+v;\beta_2+\gamma_2-1,\frac{2\lambda_2\mu_2\cos\varphi\cos\omega}{2\lambda_2\mu_2\cos\varphi\cos\omega-\{\mu_2(\lambda_2-1)e^{(\theta-\varphi-\omega)i}+\lambda_2(\mu_2-1)e^{-(\theta-\varphi-\omega)i}\}\cos\theta}\right)d\omega d\psi d\varphi d\phi d\theta,
\end{aligned}
\quad (2.15)
$$

$$
\begin{aligned}
&I_m(x;\lambda:y;\mu)I_n(u;v:v;\delta)\\
&\quad=\frac{2^{x+y+u+v+m+n}}{\pi^4}\frac{m!n!x!y!u!v!}{(m+n)!(x+u)!(y+v)!}\times e^{-\{m(\lambda+\mu)+n(v+\delta)\}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{(m-n)\psi i+(x-u)\phi i+(y-v)\theta i}\cos^{m+n}\psi\cos^{x+u}\phi\cos^{y+v}\theta\\
&\qquad\times\left\{\left(1-\frac{(1-e^{\lambda})e^{(\omega-\psi-\phi)i}+(1-e^{v})e^{-(\omega-\psi-\phi)i}}{2\cos\psi\cos\phi}\cos\omega\right)\times\left(1-\frac{(1-e^{\mu})e^{(\omega-\psi-\theta)i}+(1-e^{\delta})e^{-(\omega-\psi-\theta)i}}{2\cos\psi\cos\theta}\cos\omega\right)\right\}^{m+n}\\
&\qquad\times I_{m+n}\left[x+u;\log\left(1-\frac{(1-e^{\lambda})e^{(\omega-\psi-\phi)i}+(1-e^{v})e^{-(\omega-\psi-\phi)i}}{2\cos\psi\cos\phi}\cos\omega\right):y+v;\log\left(1-\frac{(1-e^{\mu})e^{(\omega-\psi-\theta)i}+(1-e^{\delta})e^{-(\omega-\psi-\theta)i}}{2\cos\psi\cos\theta}\cos\omega\right)\right]d\omega d\psi d\phi d\theta
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
&C_m(x;\alpha_1:y;\alpha_2)C_n(u;\beta_1:v;\beta_2)\\
&\quad=\frac{2^{x+y+u+v+m+n}}{\pi^3}\frac{m!n!x!y!u!v!}{(m+n)!(x+u)!(y+v)!}\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{(m-n)\psi i+(x-u)\phi i+(y-v)\theta i}\cos^{m+n}\psi\cos^{x+u}\phi\cos^{y+v}\theta\\
&\qquad\times C_{m+n}\left(x+u;\frac{4\alpha_1\beta_1\cos\phi\cos\psi}{\alpha_1e^{(\phi+\psi)i}+\beta_1e^{-(\phi+\psi)i}}:y+v;\frac{4\alpha_2\beta_2\cos\theta\cos\psi}{\alpha_2e^{(\theta+\psi)i}+\beta_2e^{-(\theta+\psi)i}}\right)d\psi d\phi d\theta.
\end{aligned}
\tag{2.17}
$$

References

[1] Watson GN. A note on the polynomials of Hermite and Laguerre. J Lond Math Soc 1938;13:204–9.

[2] Carlitz L. An integral for the product of two Laguerre polynomials. Boll Un Mat Ital 1962;20(3):25–8.

[3] Chatterjea SK. An integral representation for the product of two generlized Bessel polynomials. Boll Un Mat Ital 1963;18(3):377–81.

[4] Chatterjea SK. Integral representation for the product of two Jacobi polynomials. J Lond Math Soc 1964;39:753–6.

[5] Manocha HL. An integral representations for the product of two generalized Rice's polynomials. Collectanea Math 1969;20:267–70.

[6] Srivastava HM, Panda R. An integral representation for the product of two Jacobi polynomials. J Lond Math Soc 1976;12(2):419–25.

[7] Lin Shy-Der, Chao Yi Shan, Srivastava HM. Some families of hypergeometric polynomials and associated integral representations. J Math Anal Appl 2004;294:399–411.

[8] Khan MA, Khan AH, Singh Manoj. Integral representations for the product of Krawtchouk, Meixner, Charlier and Gottlieb polynomials. Int J Math Anal 2011;5(4):199–206.

Mumtaz Ahmad Khan is currently working as Professor and Chairman of Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh, India. He has to his credit 140 published and 10 accepted research papers in international journal of repute. He has successfully guided 16 Ph.D. and 11 M.Phil. students. He is member of several mathematical societies. He has evaluated more than 20 Ph.D. thesis of various other Universities. He has acted referee and reviewer of several research journals and is one of the editors of international transactions in mathematical sciences and computer.

Abdul Hakim Khan is working as Associate Professor in Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh, India from last 25 years. He has to his credit 25 published and 07 accepted research papers in national and international journal of repute. He has successfully guided 05 Ph.D. and 06 M.Phil. students. He is member of several mathematical societies.

Sayed Mohammad Abbas has secured M.Sc. with 1st division in 2005 from Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh. He did his M.Phil. in 2008 and submitted Ph.D. at Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh, India. He has published 07 papers, 03 accepted papers and 10 papers under consideration in national and international reputed journals. He has attended one international conference.